METU - NCC

LINEAR ALGEBRA MIDTERM EXAM 1	
Code : *MAT 260* Acad.Year: *2014-2015* Semester : *SPRING* Date : *22.03.2015* Time : *09:40* Duration : *90 min*	Last Name: Name : Student # : Signature : 4 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1. (15)	2. (30)	3. (25)	4. (25)				

1._(15pts)_ Let $U = \{(x, y, w, z) : -2x + 3y - w + z = 0\} \subseteq \mathbb{R}^4$. Find a basis for U. Justify your answer, that is, show that the set you find is linearly independent and it spans the subspace U.

$\mathcal{U} = \{(x, y, w, 2x-3y+w) : x, y, w \in \mathbb{R}\}$

Take $E = \{(1,0,0,2), (0,1,0,-3), (0,0,1,1)\}$

Then $\text{Span}(E) = \mathcal{U}$ because

$$(x, y, w, 2x-3y+w) = x(1,0,0,2) + y(0,1,0-3) + z(0,0,1,1)$$

Next, suppose

$$r_1(1,0,0,2) + r_2(0,1,0,-3) + r_3(0,0,1,1) = (0,0,0,0)$$

Then

$r_1 = 0$
$r_2 = 0$
$r_3 = 0$

So E is lin. ind. & E is a basis of $\mathcal{U}$.

2. *(15+15pts)* Determine if the following sets are linearly independent. Justify your answers.

(a) $E = \{x+1, \quad x^2-1, \quad x^3+x^2+x\} \subseteq \mathcal{P}_3(\mathbb{R})$.

Suppose

$$r_1(1+x) + r_2(-1+x^2) + r_3(x+x^2+x^3) = 0$$

Then

$$(r_1 - r_2) + (r_1 + r_3)x + (r_2 + r_3)x^2 + r_3 x^3 = 0$$

So

$r_3 = 0$

$r_2 + r_3 = 0 \longrightarrow r_2 = 0$

$r_1 + r_3 = 0 \longrightarrow r_1 = 0$

$r_1 - r_2 = 0$

Hence E is lin ind.

(b) $E = \{(1,1,2,-1), \quad (1,0,1,0), \quad (2,-1,1,2)\} \subseteq \mathbb{R}^4$.

Suppose

$$r_1(1,1,2,-1) + r_2(1,0,1,0) + r_3(2,-1,1,2) = (0,0,0,0).$$

Then

$r_1 + r_2 + 2r_3 = 0$

$r_1 - r_3 = 0$

$2r_1 + r_2 + r_3 = 0$

$-r_1 + 2r_3 = 0$

$r_3 = 0 \Rightarrow r_1 = 0$

$r_2 = 0$

So E is lin. indep.

3. *(25pts)* Let $E = \{2\chi_a + \chi_b - \chi_c, \quad \chi_a - 2\chi_b + \chi_c\} \subseteq \mathrm{Fun}(S)$, where $S = \{a, b, c\}$. Find the subspace $\mathrm{Span}(E)$ spanned by E.

$$\lambda \langle 2, 1, -1\rangle + \mu \langle 1, -2, 1\rangle = \langle x, y, z\rangle$$

10

$$\begin{cases} 2\lambda + \mu = x \\ \lambda - 2\mu = y \\ -\lambda + \mu = z \end{cases}$$

$$\mu = -y - z$$

$$\lambda = \mu - z = -y - 2z$$

$$-2y - 4z - y - z = x \quad \text{or}$$

10

$$x + 3y + 5z = 0$$

So, $\mathrm{Span}(E) = \{ f(a) + 3f(b) + 5f(c) = 0 \}$

5

4. *(30=10+15+5pts)* Consider subspaces $U = \{x_1 - x_2 + 2x_3 - x_5 = 0\}$ and $V = \{x_1 + x_2 - 2x_4 = 0\}$ of the vector space $\mathbb{R}^5$.

(a) Find the dimensions $\dim(U)$ and $\dim(V)$ of these subspaces. Justify your answer.

$B_U = \{\langle 1,1,0,0,0\rangle, \langle -2,0,1,0,0\rangle, \langle 0,0,0,1,0\rangle, \langle 1,0,0,0,1\rangle\}$

$\dim(U) = 4$ 5

$B_V = \{\langle 1,-1,0,0,0\rangle, \langle 2,0,0,1,0\rangle, \langle 0,0,1,0,0\rangle, \langle 0,0,0,0,1\rangle\}$

$\dim(V) = 4$ 5

(b) Find the dimensions $\dim(U \cap V)$ and $\dim(U + V)$. Justify your answer.

$U \cap V: \begin{cases} x_1 - x_2 + 2x_3 - x_5 = 0 \\ x_1 + x_2 - 2x_4 = 0 \end{cases}$ $\quad 2x_1 + 2x_3 - 2x_4 - x_5 = 0$ $\quad 2x_2 - 2x_3 - 2x_4 + x_5 = 0$

x_3, x_4, x_5 are free $\Rightarrow U \cap V =$ ~~Span(D), where~~ 10

~~$D = \{\langle -1, 0, 1, 0, 0\rangle, \langle 1, 1, 0, 0, 0\rangle\}$~~

$= \{\langle -\lambda + \mu + \frac{1}{2}\theta, \lambda + \mu - \frac{1}{2}\theta, \lambda, \mu, \theta\rangle \; ; \lambda, \mu, \theta \in \mathbb{R}\}$

$B_{U\cap V} = \{\langle -1, 1, 1, 0, 0\rangle, \langle 1, 1, 0, 1, 0\rangle, \langle \frac{1}{2}, -\frac{1}{2}, 0, 0, 1\rangle\}$

$\dim(U \cap V) = 3$

Since $\langle 1,-1,0,0,0\rangle \in V - U$, we conclude that $U + V = \mathbb{R}^5$

5

(c) Does the equality $\dim(U + V) + \dim(U \cap V) = \dim(U) + \dim(V)$ hold?

$5 + 3 = 4 + 4$

5